theory of damped oscillations of continua. Applications of Function Theory in Mechanics of a Continuous Medium. Vol. 2, "Nauka", Moscow, 1965.

12. Mikhlin, S. G., Variational Methods in Mathematical Physics. (English translation), Pergamon Press, Book № 10146, 1964.

13. Krein, S. G., Linear Differential Equations in Banach Space. "Nauka", Moscow, 1967.

Translated by M. D. F.

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ON THE NONSTATIONARY MOTION OF AN ELASTIC SPACE WITH A CRACK

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A nonstationary three-dimensional problem on the motion of an isotropic elastic medium in the presence of a crack along the half-plane is considered. Instantaneous concentrated normal and tangential pulses act at the initial instant on both edges of the half-plane. The solution for the time-periodic problem is determined by the Wiener-Hopf method, which was applied in the theory of wing vibrations [1] although the process of solving (and formulating) the problem in [1] differs from the course of the solution in this paper. Furthermore, an inverse time transformation is carried out which permits finding the solution of the nonstationary problem in the whole space at once, in the Smirnov-Sobolev form.

The problems of unsteady motion of an elastic continuous medium have been considered in [2 — 5]. The solution of a number of mixed dynamic problems for a liquid or elastic medium is given in [1, 3, 6, 7].

1. The equations of motion in displacements for an isotropic medium in the absence of body forces in the three-dimensional case are

$$\partial^2 \mathbf{v}/\partial t^2 = (a^2 - b^2)\nabla\theta + b^2\nabla^2\mathbf{v}, \quad \theta = \nabla\mathbf{v}, \quad \mathbf{v} = \{v_1, v_2, v_3\} \qquad (1.1)$$

Let us initially consider the following time-periodic singular boundary value problem for a semi-infinite slit $(z = 0, -\infty < (\infty > x$

$$\sigma_{zz} = \rho\left[(a^2 - 2b^2)\theta + 2b^2\frac{\partial v_3}{\partial z}\right] = P\delta(x + x_0)\delta(y + y_0)\exp(-i\omega t) \qquad (1.2)$$

$$y < 0, \quad v_3 = 0, \quad y > 0$$

$$\sigma_{xz} = \rho b^2\left(\frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x}\right) = Q\delta(x + x_0)\delta(y + y_0)\exp(-i\omega t)$$

$$\infty < y < \infty$$

$$\sigma_{yz} = \rho b^2\left(\frac{\partial v_2}{\partial z} + \frac{\partial v_3}{\partial y}\right) = V\delta(x + x_0)\delta(y + y_0)\exp(-i\omega t)$$

$$\infty < y < \infty$$

$$v_{1,2,3} = O(R_1^{1/2}), \quad R_1 = \sqrt{y^2 + z^2} \to 0 \quad \text{(condition on the edge)}$$

Here x_0, y_0 are positive constants, $\delta(x)$ is a delta function, ρ is the density of the elastic medium and P, Q, V are constants. Because of the boundary conditions, it is possible to consider a symmetric problem with respect to the xOy plane, where $v_{1,2}$ are even and v_3 is an odd function, and to pose the problem for the half-space $z \geqslant 0$.

Let us seek the solution of the problem (1.1), (1.2) in the form

$$v_j = v_j^\circ (x, y, z) \exp (-i\omega t) \tag{1.3}$$

$$v_i^\circ = \sum_{n=1}^{2} \iint_{-\infty}^{\infty} A_j^n \exp (i\alpha x + i\beta y + iz\gamma_n)\, d\alpha d\beta$$

$$A_1^{(2)} = -\frac{2\gamma_1\gamma_2}{C} A_1^{(1)} - \frac{i\psi_0}{C} A, \quad A_2^{(1)} = \frac{\beta}{\alpha} A_1^{(1)}, \quad A_3^{(1)} = \frac{\gamma_1}{\alpha} A_1^{(1)}$$

$$A_2^{(2)} = -\frac{2\beta}{\alpha C} \gamma_1\gamma_2 A_1^{(1)} - \frac{i\psi_0}{C} B, \quad A = \frac{Q(\gamma_2^2 - \beta^2) + V\alpha\beta}{4\pi^2 \rho b^2 \gamma_2}$$

$$A_3^{(2)} = \frac{2(\alpha^2 + \beta^2)}{\alpha C} \gamma_1 A_1^{(1)} + \frac{i(Q\alpha + V\beta)}{4\pi^2 \rho b^2 C}$$

$$B = \frac{V(\gamma_2^2 - \alpha^2) + Q\alpha\beta}{4\pi^2\rho b^2 \gamma_2}, \quad C = \gamma_2^2 - \alpha^2 - \beta^2$$

$$\gamma_n = \sqrt{k_n^2 - \alpha^2 - \beta^2}, \quad k_1^2 = \frac{\omega^2}{a^2}, \quad k_2^2 = \frac{\omega^2}{b^2}$$

$$\psi_0 = \exp (i\alpha x_0 + i\beta y_0)$$

($A_1^{(1)}(\alpha, \beta)$ is an unknown function).

It is assumed that slits along the real axis from $-\infty$ to $-\sqrt{k^2 - \alpha^2}$ and from $\sqrt{k^2 - \alpha^2}$ to ∞ are introduced for $|\alpha| < k$ (α is considered real), and $\gamma > 0$ is selected on the imaginary β-axis, while slits are made in the plane of the variable β for $|\alpha| > k$ which connect points $\pm i\sqrt{\alpha^2 - k^2}$ of the imaginary axis to the point $\pm i\infty$, respectively, and $\operatorname{Im} \gamma > 0$ is selected on the real β-axis.

The solution in the form (1.3) satisfies (1.1), the boundary conditions $\sigma_{xz}^\circ = Q\delta(x + x_0)\,\delta(y + y_0)$ and $\sigma_{yz}^\circ = V\delta(x + x_0)\,\delta(y + y_0)$ for $z = 0$. The remaining boundary conditions and the conditions on the edge determine the function $A_1^{(1)}$.

Substituting (1.3) into (1.2), we have an equation for $z = 0$, which reduces to the Wiener-Hopf equation after an inverse Fourier transformation and elimination of $A_1^{(1)}$

$$2ib^2a^{-2}(a^2 - b^2)\gamma_2^+ F^+ U^+ + f_1(\alpha, \beta) = \frac{\Omega^-}{F^- \gamma_2^-} \tag{1.4}$$

$$U^+ = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{0} (v_3^\circ)_{z=0} \exp[-i(\alpha x + \beta y)]\, dy$$

$$\Omega^- = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} \left(\frac{\sigma_{zz}^\circ}{\rho}\right)_{z=0} \exp[-i(\alpha x + \beta y)]\, dy$$

$$f_1 = \frac{i\psi_0}{\sqrt{\eta_R^2 - \alpha^2} - \beta} = f_1^+ + f_1^-, \quad f = f^{(Q,V)} +$$

$$f^{(P)} = \frac{(Q\alpha + V\beta)(C - 2\gamma_1\gamma_2) - P\gamma_1 k_2^2}{4\pi^2\rho k_2^2 \gamma_1^+ X(\alpha, \beta)}$$

$$X(\alpha, \beta) = \frac{F^-\gamma_1^-\gamma_2^-}{\sqrt{\eta_R^2 - \alpha^2} - \beta}$$

Upon factorization of the functions, the cases $|\alpha| < k$ and $|\alpha| > k$ are considered separately and it is shown that for arbitrary α

$$F(\alpha, \beta) = \frac{2}{k_2^2 - k_1^2}\left(\alpha^2 + \beta^2 + \frac{C^2}{4\gamma_1\gamma_2}\right) = \tag{1.5}$$

$$F^+(\alpha, \beta) F^-(\alpha, \beta), \quad \gamma_j = \gamma_j^+\gamma_j^-$$

$$F^\pm = \frac{\sqrt{\eta_R^2 - \alpha^2} \pm \beta}{\gamma_1^\pm \gamma_2^\pm} \exp\left[\frac{1}{2\pi i}\int_{\mp k_1}^{\mp k_2} \ln \frac{R(\eta)}{\overline{R}(\eta)} \frac{\eta d\eta}{(\sqrt{\eta^2 - \alpha^2} - \beta)\sqrt{\eta^2 - \alpha^2}}\right]$$

$$\gamma_j^\pm = (\sqrt{k_j^2 - \alpha^2} \pm \beta)^{1/2}, \quad \eta_R = \frac{\omega}{c_R}, \quad c_R < b$$

($R(\eta)$ is the Rayleigh function, η_R is the root of the function $R(\eta)$) and either the upper or the lower signs are selected simultaneously, where the functions γ_j^-, F^- and γ_j^+, F^+ are, respectively, analytic in the lower and upper half-planes of the complex variable β.

It can be shown that for any real α

$$f_1^-(\alpha, \beta) = \frac{f^{(Q,V)}(\alpha, \sqrt{\eta_R^2 - \alpha^2})}{\sqrt{\eta_R^2 - \alpha^2} - \beta}\Phi(\alpha, \eta_R) +$$

$$\int_{k_1}^{\omega\infty} \varphi(\alpha, \beta, \eta)\Phi(\alpha, \eta)\, d\eta$$

$$\varphi = \begin{cases} \varphi_1, & \eta \in (k_1, k_2) \\ \varphi_2, & \eta \in (k_2, \infty) \end{cases}$$

$$\varphi_1 = \frac{\chi(k_2^2 - 2\eta^2)}{\overline{R}(\eta)X^+(\alpha, \eta)}[(2Q\alpha + V\sqrt{\eta^2 - \alpha^2})\sqrt{k_2^2 - \eta^2} +$$

$$P(k_2^2 - 2\eta^2)], \quad \varphi_2 = \frac{\chi P}{X(\alpha, \eta)}$$

$$\chi = \frac{\eta(\sqrt{\eta^2 - \alpha^2} - \sqrt{k_1^2 - \alpha^2})^{1/2}}{4\pi^3\rho(\sqrt{\eta_R^2 - \alpha^2} - \sqrt{\eta^2 - \alpha^2})(\sqrt{\eta^2 - \alpha^2} - \beta)\sqrt{\eta^2 - \alpha^2}}$$

$$\Phi(\alpha, \eta) = \exp[i(\alpha x_0 + y_0\sqrt{\eta^2 - \alpha^2})]$$

($X^+(\alpha, \eta)$ are the upper boundary values of the function $X(\alpha, \eta)$ on the section $k_1 < \eta < k_2$).

Solving the Wiener-Hopf equation, we obtain

$$2b^2(k_2^2 - k_1^2)\gamma_2^+F^+U^+ = ik_2^2[f_1(\alpha, \beta) - f_1^-(\alpha, \beta)] \tag{1.6}$$

$$\gamma_1^-\Omega^- = X(\alpha, \beta)(\sqrt{\eta_R^2 - \alpha^2} - \beta)f_1^-(\alpha, \beta)$$

Since $U^+ = A_3^{(1)} + A_3^{(2)}$, we obtain

$$A_1^{(1)} = iA_{1,1}^{(1)}\psi_0 + iA_{2,1}^{(1)}\Phi(\alpha, \eta_R) + iA_{3,1}^{(1)}\int_{k_1}^{\omega\infty} \varphi(\alpha, \beta, \eta)\Phi(\alpha, \eta)\, d\eta \tag{1.7}$$

$$A_{1,1}^{(1)} = -\alpha\frac{PC + 2\gamma_2(Q\alpha + V\beta)}{4\pi^2\rho b^2 R(\alpha, \beta)}$$

$$A_{2,1}^{(1)} = -\alpha C X(\alpha, \beta)\frac{f^{(Q,V)}(\alpha, \sqrt{\eta_R^2 - \alpha^2})}{b^2 R(\alpha, \beta)\gamma_1^-}$$

$$b^2R(\alpha, \beta)\gamma_1^- A_{3,1}^{(1)} = -\alpha C(\sqrt{\eta_R^2 - \alpha^2} - \beta) X(\alpha, \beta)$$
$$R(\alpha, \beta) = C^2 + 4\gamma_1\gamma_2(\alpha^2 + \beta^2)$$

After evaluating the remaining coefficients by using (1.3) and (1.7), we arrive at the solution of the problem posed, which is time-periodic

$$v_j^\circ = \sum_{n,m=1}^{2} i \iint_{-\infty}^{\infty} [A_{m,j}^{(n)} \exp(i\varphi_m^{(n)}) + \frac{1}{2} A_{3,j}^{(n)} \times \tag{1.8}$$
$$\int_{k_1}^{\omega_\infty} \varphi(\alpha, \beta, \eta) \exp(i\varphi_3^{(n)})\, d\eta]\, d\alpha d\beta$$

$$\varphi_1^{(n)} = (x + x_0)\alpha + (y + y_0)\beta + z\gamma_n, \quad \varphi_2^{(n)} = (x + x_0)\alpha + y\beta + z\gamma_n + y_0\sqrt{\eta_R^2 - \alpha^2}$$
$$\varphi_3^{(n)} = (x + x_0)\alpha + y\beta + z\gamma_n + y_0\sqrt{\eta^2 - \alpha^2}$$
$$n = 1, 2; \quad j = 1, 2, 3$$

2. The inverse transform in t corresponding to the solution of the nonstationary problem for which there is $\delta(t)$ instead of $\exp(-i\omega t)$ in (1.2), is

$$v_j = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} v_j^\circ \exp(st)\, ds, \quad s = -i\omega \tag{2.1}$$

In applying the inverse Laplace transform in t, we introduce the variables $\alpha = \zeta\omega\cos\psi$, $\beta = \zeta\omega\sin\psi$ [8] in place of α, β and the polar coordinates $x + x_0 = r\cos\theta_1$, $y + y_0 = r\sin\theta_1$, $x_0 + x = \rho\cos\theta$, $y = \rho\sin\theta$ for $\varphi_1^{(n)}$ and $\varphi_{2,3}^{(n)}$, respectively.

The neighborhoods of the points $\zeta = \zeta_m^{(n)}$ for which the expressions in the exponentials vanish

$$f_m^{(n)}(\zeta_m^{(n)}) \equiv t - \varphi_m^{(n)}(\zeta_m^{(n)}) = 0 \tag{2.2}$$

are essential in the integrals with respect to ζ.

For definiteness, let us consider the first integral

$$I = -\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds \int_0^{2\pi} d\psi \int_0^{\infty} \frac{\partial}{\partial t} \{A_{1,1}^{(1)}(\zeta)\, \zeta \exp[s f_1^{(1)}(\zeta)]\}\, d\zeta \tag{2.3}$$

Here, the ω has been divided out because of homogeneity. Let us replace the quantity ψ in the integral with respect to ψ taken between $\pi < \psi < 2\pi$ by $\pi + \psi_1$. Then the coefficient of ζ in $f_1^{(1)}(\zeta)$ changes sign, where by discarding the subscript on the ψ_1, the integrals with respect to ψ and ψ_1 taken between the limits $0 < \psi < \pi$ can be combined and ζ can be replaced by $-\zeta$ in the second integral. Then

$$I = -\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds \int_0^{\pi} d\psi \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \{A_{1,1}^{(1)}(\zeta)\, \zeta \exp[s f_1^{(1)}(\zeta)]\} \operatorname{sgn}\zeta\, d\zeta \tag{2.4}$$

Let $\omega > 0$. Let us replace the contour of integration $-\infty < \zeta < \infty$ by contour Γ passing through the mentioned point $\zeta_1^{(1)}$, $\bar{\zeta}_1^{(1)}$ in the direction $\operatorname{Im} f_1^{(1)}(\zeta) = 0$. Using the notation $f_1^{(1)}(\zeta) = B_1$, where B_1 is real, $\zeta = \xi + i\mu$, it can be seen that the lines $f_1^{(1)}(\zeta) = B_1$ in the (ξ, μ) plane consist of two branches of the hyperbola

$$\frac{\xi^2}{r^2\cos^2\psi} - \frac{\mu^2}{z^2} = \frac{1}{a^2 r_1^2}, \quad r_1^2 = r^2\cos^2\psi + z^2$$

as well as of segments of the real axis $|\xi| < 1/a$.

Let $z > 0$. Then assuming that $\sqrt{a^{-2} - \zeta^2} > 0$ on the imaginary axis of the ζ-plane, it can be shown that $\operatorname{Im} f_1^{(1)}(\zeta) < 0$ in regions (see Fig. 1) where arcs of circles c_1, c_2 pass, the signs of the $\operatorname{Im} f_1^{(1)}$ are indicated in the different regions in the figure. Then the integration with respect to ζ between $-\infty$ and $r\cos\psi/(ar_1)$ is replaced by integration over the upper half of the contour Γ, while integration with respect to ζ between $r\cos\psi/(ar_1)$ and ∞ is replaced by integration over the lower half of Γ. This can be done since

$$\operatorname{Im} f_1^{(1)}(\zeta) < 0$$

on Γ_1, c_2 and it can be shown that the integrals over c_1, c_2 tend to zero as the radii of the circles c_1, c_2 increase without limit.

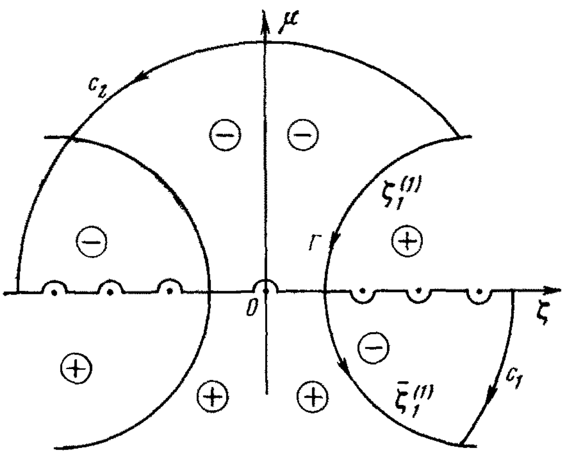

Fig. 1

Consequently, for $\omega > 0$ the integration over the real ζ-axis can be replaced by integration over Γ.

For $\omega < 0$ their complements to the upper and lower semicircles on which $\operatorname{Im} f_1^{(1)} \times (\zeta) > 0$ are taken, respectively, instead of c_1, c_2. Then integration over the real ζ-axis is replaced by integration over Γ in the direction opposite to the preceding. The inner integral in (2.4) is obtained exactly the same as for $\omega > 0$, taking $\operatorname{sgn}\zeta$ into account.

For $z < 0$ the points $\zeta_1^{(1)}$, $\bar{\zeta}_1^{(1)}$ exchange places but the solution does not change.

Thus for any ω, z we obtain from (2.4)

$$I = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds \int_0^{\pi} d\psi \int_{\Gamma} \frac{\partial}{\partial t}\{\pm\zeta A_{1,1}^{(1)}(\zeta)\exp[sf_{1,1}^{(1)}(\zeta)]\}\, d\zeta$$

where the upper and lower signs refer to the upper and lower halves of the contour Γ, which passes from top to bottom. The quantity B_1 varies between $-\infty$ and $r\cos\psi/(ar_1)$ on the upper half of Γ and between $r\cos\psi/ar_1$ to $-\infty$ on the lower half, after an inverse Laplace transformation with respect to t, we obtain by evaluating the integral of the delta function of the real argument $f_1^{(1)}(\zeta)$

$$I = -2\operatorname{Re}\frac{\partial}{\partial t}\int_0^{\pi} \frac{\zeta_1^{(1)} A_{1,1}^{(1)}(\zeta_1^{(1)})}{f_1^{(1)\prime}(\zeta_1^{(1)})}\, d\psi, \quad r_1^2\zeta_1^{(1)} = tr\cos\psi + iz\sqrt{t^2 - \frac{r_1^2}{a^2}}$$

The solution is $I = 0$ for $t < r_1/a$.

Note that the upper and lower edges of the slit $(\pm 1/a,\ r\cos\psi/r_1)$ passing from right to left and left to right, respectively, should be included in the contour Γ for $ar|\cos\psi| > r_1$ for a transverse wave with $ar > br_1$, which corresponds to the domain outside the cone passing through the line of tangency of a point and side wave. The solution is again written in the previous form in terms of roots of Eq. (2. 2), where points between the mentioned waves correspond to the mentioned slits.

The remaining integrals are calculated similarly.

The solution is finally written as follows:

$$v_j = -2\,\mathrm{Re}\,\frac{\partial}{\partial t}\sum_{n,m=1}^{2}\int_0^{\pi}\left[\frac{A_{m,j}^{(n)}(\zeta_m^{(n)})\,H(\zeta_{m,0}^{(n)})\,\zeta_m^{(n)}}{f_m^{(n)'}(\zeta_m^{(n)})} + \frac{1}{2}\int_{1/a}^{\infty}\frac{\zeta_3^{(n)}H(\zeta_0^{(n)})\,A_{3,j}^{(n)}(\zeta_3^{(n)})\,\varphi(\zeta_3^{(n)})\,d\eta}{f_3^{(n)'}(\zeta_3^{(n)})}\right]d\psi \qquad (2.5)$$

where $H(x)$ is the unit function. The relationships $\zeta_{m,0}^{(n)} = 0$, $\zeta_0^{(n)} = 0$ yield the fronts of the corresponding waves.

Note that for $x_0 = y_0 = 0$ the terms corresponding to P in the solution drop out. This is seen at once from (1. 5) which becomes

$$2ib^2a^{-2}(a^2 - b^2)\gamma_2^+F^+U^+ = \frac{P}{4\pi^2\rho F^-\gamma_2^-} + \frac{\Omega^-}{F^-\gamma_2^-} \qquad (2.6)$$

for $x_0 = y_0 = Q = V = 0$.

Hence, if it is assumed, as above, that $v_{1,2,3} = O(R_1^{1/2})$, then $U^+ \equiv 0$ is obtained, i.e. the trivial solution. Since it is clear that the solution should depend on P, it is natural to assume that the terms corresponding to P in the solution will yield $v_{1,2,3} = O(R_1^{-1/2})$ as $R_1 \to 0$; then for these terms both sides of (2. 6) should be equated to a constant determined by using the Cauchy theorem, and finally the solution for P becomes

$$U^+ = a^2K(\alpha)\,[2b^2(b^2 - a^2)F^+\gamma_2^+]^{-1}, \quad \Omega^- = K(\alpha)F^-\gamma_2^- - \frac{P}{4\pi^2}$$

$$K(\alpha) = P\,[4\pi^2F^-(\alpha, 0)\sqrt[4]{k_2^2 - \alpha^2}]^{-1}$$

Then the complete solution for $x_0 = y_0 = 0$ is given by formulas following from (1. 7) in which $x_0 = y_0 = P = 0$, and moreover, terms in $A_{2,1}^{(1)}$ should be added

$$P\alpha C\,\frac{\gamma_1^-X(\alpha, 0)\sqrt{\eta_R^2 - \alpha^2} - X(\alpha, \beta)(\sqrt{\eta_R^2 - \alpha^2} - \beta)\sqrt[4]{k_1^2 - \alpha^2}}{4\pi^2\rho b^2R(\alpha, \beta)\,X(\alpha, 0)\,\gamma_1^-\sqrt{\eta_R^2 - \alpha^2}} \qquad (2.7)$$

Note that for $x_0 = y_0 = 0$ in (2. 5) values corresponding to the range of integration $1/a < \eta < 1/b$ will be different from zero in the integrals with respect to η, where the same expressions are obtained for φ and f as above, in which $P = 0$ has been substituted.

3. In order to obtain the solution near the waves, let us apply the method of [9]. For simplicity, we consider the problem in which $x = y_0 = 0$, and we have three kinds of waves: point spherical longitudinal and transverse waves and a conical wave which is the envelope of the point transverse waves produced by the intersection between the longitudinal wave and the $z = 0$ plane.

To determine the solution near point waves with the propagation velocities c_n, where $c_1 = a$, $c_2 = b$, we introduce a new variable of integration $\sigma = \sqrt{1 - c_n^2\zeta_n^2}$ according to [9] by using (2. 2), where $x_0 = y_0 = 0$. Hence

$$\cos\psi = \frac{tc_n - z\sigma}{r\sqrt{1 - \sigma^2}}, \quad \sin\psi = \frac{R\sqrt{(\sigma - \sigma_1)(\sigma_2 - \sigma)}}{r\sqrt{1 - \sigma^2}}$$

$$R = r^2 + z^2, \quad R^2\sigma_{1,2}^{(n)} = tzc_n \pm r\sqrt{R^2 - t^2c_n^2}$$

On the fronts of the point waves $\sigma_*^{(n)} \equiv \sigma_1^{(n)} = \sigma_2^{(n)} = tzc_nR^{-2}$, and we replace the contour of integration in the σ-plane by a small segment connecting the points $\sigma_1^{(n)}$, $\sigma_2^{(n)}$ [9] in determining the solution near the waves by the Cauchy theorem and by evaluating the integral, we obtain near the point waves

$$v_j = \frac{2\pi tz}{R^2} \sum_{n=1}^{2} \delta\left(t - \frac{R}{c_n}\right) \times \tag{3.1}$$

$$[B_j^{(n)}(\sigma) + \int_{1/a}^{1/b} A_{3,j}^{(n)}(\sigma)\varphi(\sigma, \eta)\, d\eta]_{\sigma=\sigma_*^{(n)}}, \quad B_j^{(n)}(\sigma) = \sum_{m=1}^{2} A_{m,j}^{(n)}(\sigma)$$

where $A_{m,j}^{(n)}$ are given by (1.7) taking (2.7) into account.

To determine the solution near the conical wave with the equation

$$t = t_f = (r\sin\beta + z\cos\beta)/b, \quad \cos\beta = \gamma = \sqrt{a^2 - b^2}/a$$

(β is an angle measured from the z-axis in the vertical plane, corresponding to the tangent circle of the conical and point transverse waves), the relative location of the points $\sigma = \sigma_1$ and $\sigma = \gamma$ on the real σ-axis should be studied. For points $M(x, y, z)$ ahead of the conical wave we have $\gamma < \sigma_1$, i.e. a slit in the σ-plane corresponding to branch points γ, σ_1 of the integrand lies outside the contour of integration. For points $M(x, y, z)$ behind the conical wave, the value of the integral in the σ-plane can be replaced by integrals over the edges of the slit (σ_1, γ), where $\sigma_1 < \gamma$. Then we obtain the solution near the conical wave

$$v_j = \frac{2\pi}{b\sqrt{r}}\left(\frac{\cos\beta}{l}\right)^{3/2} \sin\beta \times \tag{3.2}$$

$$\frac{\partial}{\partial t}\left\{\operatorname{Im}\left[A_{3,j}^{(2)}(\sigma)\int_{1/a}^{1/b} \varphi(\sigma, \eta)\, d\eta + B_j^{(2)}(\sigma)\right]_{\sigma=\gamma} \Delta n + O[(\Delta n)^2]\right\}$$

$$l = r\cos\beta - z\sin\beta, \quad \Delta n \sin\beta = b(t_f - t), \quad bt_f = r\sin\beta + z\cos\beta$$

Near the tangent line of the waves $l \approx 0$ and (3.2) is not applicable, but as in [9], the solution can be obtained in the neighborhood mentioned.

Therefore, the solutions (3.1) and (3.2) near the point and conical waves have been obtained from the general formulas (2.5) in the case when the force is concentrated at the point (0, 0, 0).

The principal terms in the stress intensity coefficient and in the value of the displacement v_3 near $z = 0$, $y = \pm 0$ can be obtained from (2.5) — (2.7), respectively, as

$$\sigma_{zz} = \frac{P}{2r\pi^2}\operatorname{Re}\frac{\partial^2}{\partial t^2}\left(\frac{T}{|\theta|}\right)^{3/2}\int_0^\infty \frac{\mu_+ + i\mu_-}{\lambda^2}\, d\lambda$$

$$v_3 = \frac{Pa^2}{4\pi^2 b^2\rho(a^2 - b^2)r}\operatorname{Re}\frac{\partial}{\partial t}\sqrt{\frac{T}{|\theta|}}\int_0^\infty \frac{\mu_- + i\mu_+}{\lambda}\, d\lambda$$

$$\mu_\pm = [(b^{-2} - T^2)\lambda^2 - T^2(1 \pm 2\lambda)]^{-1/4}$$

$$T = tr^{-1}, \quad x = r\cos\theta, \quad y = r\sin\theta$$

We have $T \gg 1$ near the point $x = y = z = 0$ and we obtain

$$\sigma_{zz} = \frac{P\sqrt{2}}{6\pi^2 b^2 t^3 |\theta| \sqrt{|\theta|}}, \quad v_3 = \frac{Pa^2\sqrt{2}}{8\pi\rho b^2 (a^2 - b^2) tr \sqrt{|\theta|}}$$

Note that the fundamental part of the singularity near the edge corresponding to P has been extracted, where the terms in σ_{zz} and v_3 corresponding to the transverse loads Q and V yield the singularity $|\theta|^{-1/2}$ and $|\theta|^{1/2}$, respectively, i.e. the smoother terms in the solution.

Let us study the domain near the front for waves reflected from the edge $z = y = 0$ in the general problem in which $x_0 \neq 0$, $y_0 \neq 0$. The terms containing $A_{1,j}^{(n)}$ in the solution (2.5) correspond to incident longitudinal and transverse waves, i.e. to the Lamb solution for the half-space. The solution for the neighborhoods of the mentioned waves are found by calculations similar to the calculations to obtain (3.2), and has the same singularity in the form of a δ-function.

It is more complicated to obtain the solution for waves reflected from the slit edges, to which the remaining terms in (1.8) correspond, where $y_0\sqrt{\eta^2 - \alpha^2}$ is the time at which perturbations given by the integrands in the integral with respect to η originate at the point $(-x_0, 0)$. Hence η has the meaning of velocities of perturbations arriving at the point $(-x_0, 0)$ and generating reflected waves. Therefore, it is necessary to determine the behavior of terms containing integrals with respect to η in (2.5) in the neighborhood of the waves. We note that the fronts of waves reflected from the edge can be obtained from the equation of the envelope of "plane" waves

$$f_3^{(n)}(\zeta_3^{(n)}) = \frac{\partial}{\partial \zeta_3^{(n)}} f_3^{(n)}(\zeta_3^{(n)}) = \frac{\partial}{\partial \psi} f_3^{(n)}(\zeta_3^{(n)}) = 0 \tag{3.3}$$

Henceforth, the subscript 3 is discarded for brevity, $\eta = 1/a$ corresponds to waves generated by an incident longitudinal wave, and $\eta = 1/b$ by transverse wave. Note that two waves (longitudinal and transverse), produced by longitudinal and transverse waves incident on the edge, correspond to the mentioned integrals with respect to η, which is due to the discontinuous nature of φ_η' and it is possible to write in (2.5)

$$\int_{1/a}^{\infty} \lambda_{n,j}\varphi(\zeta_3^{(n)}, \eta)\, d\eta = \sum_{k=1}^{2} \int_{1/c_k}^{\infty} \lambda_{n,j}\nu_k d\eta \tag{3.4}$$

$$\lambda_{n,j} = \zeta_3^{(n)} H(\zeta_0^{(n)})\, A_{3,j}^{(n)}(\zeta_3^{(n)})\, [f_3^{(n)\prime}(\zeta_3^{(n)})]^{-1}, \quad \nu_1 = \varphi_1, \quad \nu_2 = \varphi_2 - \varphi_1$$

Let us first consider terms corresponding to the first integral in the right side of (3.4). Let $\psi_0^{(n)}$, $\zeta_0^{(n)}$ denote the quantities ψ, ζ given by the mentioned wave equations $(\eta = 1/a)$. We introduce the variable $\sigma = \sqrt{1 - c_n^2\zeta^2}$. It is possible to consider $\psi_0 < \psi < \psi_0 + 2\pi$ for the limits of integration in polar coordinates in the solution (1.8) and $\psi_0 < \psi < \psi_0 + \pi$ in (2.5). Then, repeating the discussion carried out in obtaining (3.1), we obtain for the mentioned terms in (2.5) near the waves

$$-2\,\mathrm{Re} \sum_{n=1}^{2} \frac{\partial}{\partial t} \int_0^{\infty} d\mu \int_{\sigma_1^{(n)}}^{\sigma_2^{(n)}} \frac{\sigma A_{3,j}^{(n)}(\zeta^{(n)})\,\varphi_1(\zeta^{(n)}, \eta)\, H(f^{(n)})}{c_n^2\,\partial f^{(n)}/\partial\psi}\, d\sigma \tag{3.5}$$

$$\mu = \sqrt{\eta^2 - \alpha^2} - \sqrt{a^{-2} - \alpha^2}$$

We discard the superscript n in the intermediate calculations, then we can write $\sigma_1 = \sigma(\psi_0)$, $\sigma_2 = \sigma(\pi + \psi_0)$ and $\zeta(\psi)$ is given by the equation $f = 0$. Hence, it is seen that

$$\zeta(\psi_0) = -\bar{\zeta}(\pi + \psi_0), \quad \sigma(\psi_0) = \bar{\sigma}(\pi + \psi_0)$$

where the bar denotes the complex-conjugate value and $\sigma_{1,2} = \sigma_0$, $c\zeta_0 = \sqrt{1 - \sigma_0^2}$. on the wave.

Near the wave we can assume

$$\frac{\partial f}{\partial \psi} \approx \frac{\partial^2 f}{\partial \psi^2}(\psi - \psi_0) + \frac{\partial^2 f}{\partial \psi \partial \sigma}(\sigma - \sigma_0)$$

According to equation $f(\sigma, \psi) = 0$ we have

$$\frac{\partial^2 f}{\partial \psi^2}(\psi - \psi_0) \approx -\frac{\partial^2 f}{\partial \psi \partial \sigma}(\sigma - \sigma_0) + \left[-K_0(\sigma - \sigma_0)^2 - 2\frac{\partial^2 f}{\partial \psi^2}(t - t_f - y_0\mu)\right]^{1/2}$$

$$K_0 = \frac{\partial^2 f}{\partial \psi_2}\frac{\partial^2 f}{\partial \sigma^2} - \left(\frac{\partial^2 f}{\partial \psi \partial \sigma}\right)^2, \quad t_f = \zeta_0\rho\cos(\theta - \psi_0) + z\sqrt{c^2 - \zeta_0^2} +$$
$$y_0\sqrt{c^2 - \zeta_0^2\cos^2\psi_0}$$

Here $t = t_f$ is the equation of the wave (3.3) for $\eta = a^{-1}$, $\partial^2 f / \partial\psi^2 > 0$, $K_0 > 0$. It can be shown that

$$\sigma_{1,2} = \sigma_0 \pm i\left[\frac{2}{K_0}\frac{\partial^2 f}{\partial \psi^2}(t - t_f - y_0\mu)\right]^{1/2}$$

Evaluating the integral in (3.5) and repeating the discussion presented above for the second terms in the right side of (3.4), using the notation $\bar{\mu} = \sqrt{\eta^2 - \alpha^2} - \sqrt{b^{-2} - \alpha^2}$ near the waves reflected from the edge, we obtain

$$-2\pi \operatorname{Re} \sum_{n=1}^{2} \frac{1}{y_0 r_n^2 \sqrt{K_0^{(n)}}} [\sigma_0^{(n)} H(t - t_f^{(n)}) A_{3,j}^{(n)}(\zeta_0^{(n)}) \nu_1(\zeta_0^{(n)}, a^{-1}) + \quad (3.6)$$
$$\bar{\sigma}_0^{(n)} H(t - \bar{t}_f^{(n)}) A_{3,j}^{(n)}(\bar{\zeta}_0^{(n)}) \nu_2(\bar{\zeta}_0^{(n)}, b^{-1})]$$

Here $\bar{\zeta}_0^{(n)}$ are values of $\zeta^{(n)}$ in (3.3) on the fronts of waves generated by an incident transverse wave on which $\eta = b^{-1}$ and $\bar{t}_f^{(n)}$ are equations of the mentioned wave fronts; $\mu_0^{(n)} = (t - t_f^{(n)})y_0^{-1} \approx 0$, $\bar{\mu}_0^{(n)} = (t - \bar{t}_f^{(n)})y_0^{-1} \approx 0$ near the appropriate wave fronts.

In obtaining the last formulas yielding the solution near the waves reflected from an edge it has been taken into account that $\mu_0^{(n)}$, $\bar{\mu}_0^{(n)}$ should replace the upper limit ∞ in the integrals with respect to μ since for $t < t_f^{(n)}$, $t < \bar{t}_f^{(n)}$ there are no reflected waves at this point.

As is seen from (3.6), the solution near the reflected waves is smoother than a solution of the δ-function kind behind the incident waves. The exception is the solution near waves reflected from the edge during Rayleigh wave incidence and given by $f_2^{(n)}$ in (2.5), behind which the singularity in the solution is again a δ-function.

It should be noted that for $x_0 = y_0 = 0$ the inversion of the Laplace transform can be obtained also by the method in [10].

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REFERENCES

1. Miles, J. W., The oscillating rectangular airfoil at supersonic speeds. Quart. Appl. Math., Vol. 9, № 1, 1951.

2. Babich, V. M., Kapilevich, M. B., Mikhlin, S. G., Natanson, G. I.,

Giz, P. M., Slobodetskii, L. N. and Smirnov, M. M., Linear Equations of Mathematical Physics. "Nauka", Moscow, 1964.
3. Cherepanov, G. P., Elastic wave diffraction by a slit. Mechanics of Continuous Media and Kindred Problems of Analysis (on the 80-th Birthday of Academician N. I. Muskhelishvili). "Nauka", Moscow, 1972.
4. Frank, P. and Mises, R., Differential and Integral Equations of Mathematical Physics. ONTI, Moscow-Leningrad, 1937.
5. Nigul, U. K., Deviations of the solution of a quasilinear wave equation from the solution of a linear equation in the domain of continuous first derivative. PMM Vol. 37, № 3, 1967.
6. Zvolinskii, N. V., Flitman, L. M., Kostrov, B. V. and Afanas'ev, V. A., Some problems of elastic wave diffraction. Applications of Function Theory to Problems of the Mechanics of a Continuous Medium. "Nauka", Moscow, 1965.
7. Sveklo, V. A., The mixed problem for the elastic anisotropic half-plane. PMM Vol. 26, № 5, 1962.
8. Bagdoev, A. G. and Martirosian, A. N., Solution of a number of nonstationary three-dimensional problems for a continuous medium in the presence of concentrated pulses. Izv. Akad. Nauk ArmSSR, Mekhanika, Vol. 27, № 3, 1974.
9. Zvolinskii, N. V., Reflected and head waves originating on the interface of two elastic media. Izv. Akad. Nauk SSSR, Ser. Geofiz., № 10, 1957.
10. Cagnard, L., Réflexion at Réfraction des Ondes Seismiques Progressives. Gauthier-Villard, Paris, 1939.

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CONTACT PROBLEM FOR A STAMP WITH A RECTANGULAR BASE

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A method is proposed to solve the problem of impressing a rectangular stamp with arbitrary ratio between the sides into an elastic isotropic half-space, based on reduction of the problem to two-dimensional dual integral equations. A method of reducing these equations to an infinite system of linear algebraic equations is indicated. Formulas are obtained to determine the pressure on the contact area and the displacement of the stamp.

The papers [1 — 4] have been devoted to contact problems for a rectangular stamp. The problem of impressing a stamp with a base in the form of a narrow rectangle into an elastic half-space has been studied in [5 — 7].

The method of solution used in this paper is a further development and extension (to the case of two-dimensional dual equations) of the method used in [7].

1. We use a rectangular x, y, z coordinate system whose z-axis is perpendicular to the boundary of the half-space. Let a stamp with a rectangular planform (Fig. 1) be